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BOUNDARY VALUE PROBLEMS IN THIN SHALLOW SHELLS OF ARBITRARY PLAN FORM

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SUMMARY

This paper is concerned with a general method of analysis of boundary-value problems in thin shallow shells of arbitrary plan form. Two specific shell configurations are considered. General solutions to the governing partial differential equations are obtained in complex form, containing a sufficient number of arbitrary elements to satisfy the four boundary conditions permitted by classical thin shell theory. An algorithm for the determination of these arbitrary elements from a general form of boundary condition is presented. The method of solution is based on I.N. Vekua's theory of elliptic partial differential equations.

Part 1 of the paper is devoted to shallow spherical shells. An example calculation is given for a circular planform shell, for which a closed form of solution may be obtained. The computed results show close agreement with the exact solution.

Part 2 of the paper deals with shallow circular cylindrical shells, including the calculation of a shell the planform of which is a square with rounded corners. Graphs of deflection and stress function are given.

1. *Introduction.*

The solution of boundary value problems of shallow shells has been the subject of many papers during recent years. The plan form of the shell and the particular boundary conditions to be applied have proved to be a major influence in the choice of a method of solving the governing partial differential equations. This has resulted in a large number of solution methods becoming available for shells of either rectangular or circular plan form. Most of these methods are suitable only for particular types of boundary condition, for example a simply supported shell of rectangular plan form may be analysed by the method of double Fourier-series expansion, but this method does not apply to shells having any other type of boundary conditions or plan form. A major difficulty has been the necessity to reduce the governing partial differential equations to ordinary differential equations, for which many general methods of solution are available (e.g. [4]). The analysis of shells of arbitrary plan form has received little attention, precisely because of this difficulty.

This paper presents a method of analysis of shallow shells of arbitrary plan form, and subject to a general form of boundary condition. The shells are assumed to be isotropic and elastic and to undergo small deflections only. The analysis of each shell then reduces to the solution of the well known shallow shell equations [1], subject to the appropriate boundary conditions. This system of partial differential equations, in cartesian co-ordinates (x, y) is written as a single matrix equation. The solution of this reduces to the consideration of two subsidiary second order matrix equations. These equations are then solved by using the very extensive theory of elliptic partial differential equations developed by I. N. Vekua in [2]. This theory is based on the reduction of the elliptic equations to equations of the hyperbolic type by the introduction of new complex independent variables. General solutions are then set up in terms of a special function of the basic equation, called the Riemann Function and certain arbitrary analytic complex functions. These arbitrary functions are specified by satisfaction of the shell boundary conditions. Complex variable theory permits the arbitrary functions to be represented as Cauchy-type contour integrals, and their determination from the boundary conditions reduces to the solution of a real singular integral equation.

Two examples are given, the first of which provides a verification of the numerical procedure, and the second shows how the technique can be applied to hitherto unsolved problems. The first case is that of a shallow spherical shell, supported on a meridional circle and subjected to a uniformly distributed load. This boundary value problem is then converted, using the method described, into a real singular integral equation. This integral equation is solved numerically and the solution compared with the exact solution. The second example is a boundary value problem in a shallow circular cylindrical shell of less simple plan form.

The shell boundaries are assumed smooth, but no additional restrictions are imposed, other than that shallow shell theory must be applicable. The analysis of various non-symmetric plan forms of shallow shells therefore becomes possible by a direct application of the method outlined in the following paragraphs. A notable feature of the analysis is that it does not involve the reduction of partial differential equations to ordinary differential equations.

PART 1

1. *General Solution of Boundary Value Problems for Shallow Spherical Shells of Arbitrary Plan Form.*

The partial differential equations governing the behaviour of the linearly elastic and isotropic shallow spherical shell are given by Novozhilov [1] as:

$$\left.\begin{aligned} &\Delta\,\Delta\,(\Phi) - \frac{Eh}{r}\,\Delta(w) = 0, \\ &\frac{E h^3}{12(1-\nu^2)}\,\Delta\,\Delta(w) + \frac{1}{r}\,\Delta(\Phi) = q. \end{aligned}\right\} \qquad (1.1)$$

The radial displacement w and stress function Φ define the stress resultants according to expressions given in [1]. The loading q is assumed to be normal to the surface. The shell thickness, h, the Young's Modulus of the shell material, E, and the Poisson's Ratio of the material ν are con-considered to be constant. Δ = Laplace Operator, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

The equations 1.1, are more conveniently expressed by defining new variables according to the following formulae:

$$\overline{\Phi} = \frac{\sqrt{12(1-\nu^2)}}{r^2 h}\,\Phi, \quad \overline{w} = \frac{w}{r^2/Eh}. \qquad (1.2)$$

Substitution of the new variables $\overline{\Phi}$ and $\overline{w}$ into the equations 1.1. yields the following system:

$$\left.\begin{aligned} &\Delta\,\Delta\,(\overline{\Phi}) - \alpha^2\,\Delta(\overline{w}) = 0, \\ &\Delta\,\Delta\,(\overline{w}) + \alpha^2\,\Delta(\overline{\Phi}) = \alpha^4 q, \end{aligned}\right\} \qquad (1.3)$$

in which α is a convenient shell parameter defined by:

$$\alpha^2 = \frac{\sqrt{12(1-\nu^2)}}{rh}. \tag{1.4}$$

The L. H. S. of the system 1.3 may be factorised and the equations written in matrix form as:

$$\Delta \left(\Delta \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha^2 \\ \alpha^2 & 0 \end{bmatrix} \right) \begin{Bmatrix} \overline{\Phi} \\ \overline{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \alpha^4 q \end{Bmatrix} \tag{1.5}$$

The general solution of the matrix partial differential equation 1.5 may conveniently be written as:

$$\begin{Bmatrix} \overline{\Phi} \\ \overline{w} \end{Bmatrix} = \begin{Bmatrix} V_0 \end{Bmatrix} + \begin{Bmatrix} V_1 \end{Bmatrix} + \begin{Bmatrix} V_2 \end{Bmatrix}, \tag{1.6}$$

Where V_0, V_1, V_2 are (2 x 1) vectors satisfying the conditions:

1. V_0 is any solution of 1.5;
2. V_1 is a solution of the homogeneous Laplace equation

$$\Delta \begin{Bmatrix} V_1 \end{Bmatrix} = 0 \tag{1.7}$$

3. V_2 is a solution of the homogeneous equation:

$$\left(\Delta \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha^2 \\ \alpha^2 & 0 \end{bmatrix} \right) \begin{Bmatrix} V_2 \end{Bmatrix} = 0 \tag{1.8}$$

The shallow spherical shell equations may therefore be solved from the consideration of two subsidiary second-order systems 1.7 and 1.8.

The theory of elliptic partial differential equations of the form 1.8 has been extensively developed by I. N. Vekua [2]. This theory is of great importance to the applied mathematician since it permits the construction of solutions to a wide variety of boundary value problems. The solution of equation 1.8 (and 1.7 as a particular case of 1.8 with $\alpha = 0$) is obtained by a direct application of Vekua's theory as given in [2], [3].

New independent variables z, ζ are defined as:

$$z = x + iy, \quad \zeta = x - iy. \tag{1.9}$$

The equation 1.8 may therefore be transformed from an elliptic equation to an equation of the hyperbolic type by substitution of the new independent variables z, ζ, i.e. 1.8 becomes:

$$\frac{\partial^2}{\partial z\, \partial \zeta} \begin{Bmatrix} V_2 \end{Bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & -\alpha^2 \\ \alpha^2 & 0 \end{bmatrix} \begin{Bmatrix} V_2 \end{Bmatrix} = 0 \tag{1.10}$$

A special function of the equation 1.10 is now introduced, this is dependent only on the form of equation 1.10 and is independent of the particular domain or boundary conditions under consideration. This function is named the Riemann Function because of its formal analogy with Riemann's classical method of integrating hyperbolic differential equations. The Riemann Function $G(z, \zeta, t, \tau)$ for equation 1.10 is defined as the solution of the following Volterra integral equation:

$$\begin{bmatrix} G(z,\zeta,t,\tau) \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & -\alpha^2 \\ \alpha^2 & 0 \end{bmatrix} \int_t^z \int_\tau^\zeta \begin{bmatrix} G(t_1,\tau_1,t,\tau) \end{bmatrix} dt_1\, d\tau_1 = I_2, \tag{1.11}$$

where $t = x_0 + iy_0$, $\tau = x_0 - iy_0$ are fixed points within the fundamental domain of analyticity of the coefficients of equation 1.10 (which in the case α = constant is the whole complex plane); I_2 is the unit matrix of order 2.

Solving equation 1.11 by the method of successive approximations, the Riemann Function is expressed explicitly in terms of the Bessel Function of the first kind and order zero:

$$\left[G(z, \zeta, t, \tau)\right] = J_0\left(\sqrt{\begin{bmatrix} 0 & -\alpha^2 \\ \alpha^2 & 0 \end{bmatrix}^{(z-t)(\zeta-\tau)}}\right) \tag{1.12}$$

The Riemann Function of equation 1.10 may therefore be expressed as a uniformly convergent series. This series is conveniently computed for arbitrary values of the parameters by using the Cayley-Hamilton theorem to evaluate the powers of the (2 x 2) matrix appearing in the argument of J_0.

The Riemann Function corresponding to equation 1.7 is obtained by putting $\alpha = 0$ in the relationships 1.11 and 1.12 i.e. it is the unit matrix of order 2.

The general solution of equation 1.10, and equation 1.8 from which it is derived, in an arbitrary simply-connected domain, is given by Vekua as:

$$\left\{V_2(z)\right\} = \mathrm{Re}\left(\left[G(z,o,z,\bar{z})\right]\left\{\phi_2(z)\right\} - \int_0^z \frac{\partial}{\partial\sigma}\left[G(\sigma,o,z,\bar{z})\right]\left\{\phi_2(\sigma)\right\}d\sigma\right), \tag{1.13}$$

where $\phi_2(z)$ is an arbitrary analytic complex vector of order 2. The general solution of the Laplace Equation 1.7 is obtained from 1.13 by putting α = 0, i.e.

$$\left\{V_1(z)\right\} = \mathrm{Re}\left\{\phi_1(z)\right\} \tag{1.14}$$

where $\phi_1(z)$ is an arbitrary analytic complex vector of order 2.

The solution of the shallow spherical shell equations can be represented in the following form using 1.13, 1.14 and 1.6:

$$\begin{Bmatrix} \bar{\bar{\Phi}}(z) \\ \bar{w}(z) \end{Bmatrix} = \left\{V_0(z)\right\} + \mathrm{Re}\left(\left\{\phi_1(z)\right\} + \left[G(z, o, z, \bar{z})\right]\left\{\phi_2(z)\right\} - \int_0^z \frac{\partial}{\partial\sigma}\left[G(\sigma, o, z, \bar{z})\right]\left\{\phi_2(\sigma)\right\}d\sigma\right). \tag{1.15}$$

Equations 1.15 are more conveniently written as:

$$\begin{Bmatrix} \bar{\bar{\Phi}}(z) \\ \bar{w}(z) \end{Bmatrix} = \left\{V_0(z)\right\} + \mathrm{Re}\left(\begin{bmatrix} +1 & 0 & \\ & & G(z, 0, z, \bar{z}) \\ 0 & +1 & \end{bmatrix}\left\{\phi(z)\right\} - \int_0^z \frac{\partial}{\partial\sigma}\begin{bmatrix} +1 & 0 & \\ & & G(\sigma, 0, z, \bar{z}) \\ 0 & +1 & \end{bmatrix}\left\{\phi(\sigma)\right\}d\sigma\right) \tag{1.16}$$

ϕ is now an arbitrary complex vector of order 4, and is determined by satisfaction of the physical boundary conditions prevailing at the edge of the shell. Classical thin shell theory admits 4 boundary conditions which must be satisfied at each point on the shell perimeter. The most general form of shell boundary conditions may be written:

$$\sum_{j,k=0}^{j+k\leq n}\left[A_{j,k}(t_0)\right]\frac{\partial^{j+k}}{\partial x^j \partial y^k}\begin{Bmatrix} \bar{\bar{\Phi}} \\ \bar{w} \end{Bmatrix} = \left\{f_0(t_0)\right\}, \tag{1.17}$$

where t_0 is any point on the shell perimeter:

$A_{j,k}$ is a (4 x 2) matrix, and f_0 is a (4 x 1) vector, which is zero in the case of homogeneous boundary conditions. The arbitrary complex vector ϕ is now represented by a Cauchy-type contour integral around the shell boundary as follows:

$$\left\{\phi(z)\right\} = \int_L \left(1-\frac{z}{t}\right)^{n-1} \lg\left(1-\frac{z}{t}\right)\left\{\mu(t)\right\} ds + \int_L \left\{\mu(t)\right\} ds, \tag{1.18}$$

Where t is any point on the smooth shell boundary L; ds is an element of length of L; and μ is an arbitrary real function of points t on L. The notation lg implies that the principal value of the logarithm is to be taken. It should be noted that the order of the highest derivative of the unknown in the boundary conditions 1.17 i.e. 'n', also appears in the integral representation 1.18. It is apparent therefore that derivatives of the R.H.S. integrand of order (n - 2) are continuous; derivatives of order (n - 1) have a singularity of the logarithmic type, and nth derivaties have a singularity of the Cauchy type. Substituting for ϕ from 1.18 into 1.16 yields the final representation of the solution of the shallow spherical shell equations as follows:

$$\begin{Bmatrix} \bar{\Phi}(z) \\ \bar{w}(z) \end{Bmatrix} = \left\{V_0(z)\right\} + \int_L \left[K_0(z,t)\right]\left\{\mu(t)\right\} ds, \tag{1.19}$$

where $K_0(z,t)$ is a (2 x 4) matrix given by:

$$K_0(z,t) = \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} G(o,o,z,\bar{z}) + \mathrm{Re}\left(\begin{bmatrix} +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +1 \end{bmatrix}\right)\left(1-\frac{z}{t}\right)^{n-1}\lg\left(1-\frac{z}{t}\right)$$
$$- \int^{z} \frac{\partial}{\partial\sigma}\left[\begin{matrix} +1 & 0 \\ 0 & +1 \end{matrix}\; G(\sigma,o,z,\bar{z})\right]\left(1-\frac{\sigma}{t}\right)^{n-1}\lg\left(1-\frac{\sigma}{t}\right) d\sigma\Bigg) \tag{1.20}$$

Substitution of the general solution from 1.19 into the boundary conditions 1.17 yields a real singular integral equation for the unknown real vector μ as follows:

$$\left[A(t_0)\right]\left\{\mu(t_0)\right\} + \int_L \left[K(t_0,t)\right]\left\{\mu(t)\right\} ds = \left\{f(t_0)\right\}, \tag{1.21}$$

where $A(t_0)$, $K(t_0, t)$ are (4 x 4) square matrices defined by:

$$A(t_0) = \mathrm{Re}\left(\frac{\pi i(-1)^n(n-1)!}{t_0^{n-1}\left(\frac{dt}{ds}\right)_{t=t_0}} \times \sum_{j=0}^{n} i^j \left[A_{n-j,j}(t_0)\right]\begin{bmatrix} +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +1 \end{bmatrix}\right),$$

$$K(t_0,t) = \sum_{j,k=0}^{j+k\leq n}\left[A_{j,k}(t_0)\right]\frac{\partial^{j+k}}{\partial\xi^j\,\partial\eta^k}\left[K_0(t_0,t)\right], \quad (t_0 = \xi + i\eta) \tag{1.22}$$

and $f(t_0)$ is a (4 x 1) vector given by

$$\left\{f(t_0)\right\} = \left\{f_0(t_0)\right\} - \sum_{j,k=0}^{j+k\leq n}\left[A_{j,k}(t_0)\right]\frac{\partial^{j+k}}{\partial\xi^j\,\partial\eta^k}\left\{V_0(t_0)\right\}.$$

The solution for boundary-value problems for shallow spherical shells in simply-connected domains may therefore be reduced to the solution of a real singular integral equation of the type 1.19. The analytical solution of such equations is often very difficult, but a solution may be found easily and simply by numerical approximation.

2. *Numerical Example of a Shallow Spherical Shell Supported Around a Meridional Circle Under Uniform Loading*

Consider a shallow spherical shell of radius 7.25 inches, thickness 0.0625 inches, and supported on a meridional circle of radius a = 2.75 inches. Let the Poisson's ratio, ν, of the material be 0.3. These dimensions refer to a shell previously studied for the effect of elliptical discontinuities [5].

The boundary conditions at the edge of the shell are assumed to be:

$$\overline{w} = \frac{\partial^2 \overline{w}}{\partial r^2} = \overline{\overline{\Phi}} = \frac{\partial^2 \overline{\overline{\Phi}}}{\partial r^2} = 0, \qquad (1.23)$$

where r, θ, are polar co-ordinates of the shell plan.

The general solution for the shallow spherical shell 1.16 is written in terms of a complex co-ordinate z = x + iy, where x and y refer to a system of cartesian co-ordinates. The boundary conditions 1.23 in co-ordinates (x, y) are:

$$\left.\begin{aligned} \overline{w} &= \cos^2\theta \frac{\partial^2 \overline{w}}{\partial x^2} + \sin 2\theta \frac{\partial^2 \overline{w}}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 \overline{w}}{\partial y^2} = 0, \\ \overline{\overline{\Phi}} &= \cos^2\theta \frac{\partial^2 \overline{\overline{\Phi}}}{\partial x^2} + \sin 2\theta \frac{\partial^2 \overline{\overline{\Phi}}}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 \overline{\overline{\Phi}}}{\partial y^2} = 0, \end{aligned}\right\} \qquad (1.24)$$

The general boundary conditions 1.17 specialised to describe the conditions given by equation 1.24, become:

$$\begin{bmatrix} +1 & 0 \\ 0 & +1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{bmatrix} \frac{\partial^2}{\partial x^2} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sin 2\theta & 0 \\ 0 & \sin 2\theta \end{bmatrix} \frac{\partial^2}{\partial x \partial y} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} \frac{\partial^2}{\partial y^2} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (1.25)$$

Using the $A_{i,j}$ notation, conditions 1.25 become:

$$A_{0,0} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} + A_{2,0} \frac{\partial^2}{\partial x^2} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} + A_{1,1} \frac{\partial^2}{\partial x \partial y} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} + A_{0,2} \frac{\partial^2}{\partial y^2} \begin{Bmatrix} \overline{\overline{\Phi}} \\ \overline{w} \end{Bmatrix} = 0. \qquad (1.26)$$

The order of the highest derivatives of the unknowns $\overline{\overline{\Phi}}$ and $\overline{w}$ appearing in the boundary conditions is 2, therefore the arbitrary (4 x 1) complex vector ϕ of the general solution 1.16 may be represented as a contour integral by the following identity:

$$\left\{ (z) \right\} = \int_L \left(1-\frac{z}{t}\right) \lg \left(1-\frac{z}{t}\right) \left\{\mu(t)\right\} ds + \int_L \left\{\mu(t)\right\} ds, \qquad (1.27)$$

The contour L in this example being a circle, with centre at the origin of co-ordinates, and of radius 2.75 inches. Substitution of the integral representation of ϕ into the general solution 1.16 yields a formula of the type 1.19 i.e.:

$$\begin{Bmatrix} \bar{\bar{\Phi}}(z) \\ \bar{w}(z) \end{Bmatrix} = \Big\{ V_0(z) \Big\} + \int_L \Big[K_0(z,t) \Big] \Big\{ \mu(t) \Big\} ds, \tag{1.28}$$

Where $K_0(z, t)$ is defined by 1.20 with n = 2. Substitution of the general solution 1.28 into the boundary conditions 1.26 yields a real singular integral equation of the type 1.21.

$$\Big[A(t_0)\Big] \Big\{\mu(t_0)\Big\} + \int_L \Big[K(t_0,t)\Big] \Big\{\mu(t)\Big\} ds = \Big\{f(t_0)\Big\}, (t_0 = \xi + i\eta,\ t_0 \epsilon L) \tag{1.29}$$

In this case the kernel of the integral equation has only a singularity of the Cauchy-type, and does not have logarithmic singularities, since derivatives of order (n - 1) do not appear in the boundary conditions.

The choice of a particular solution vector V_0 is quite arbitrary; however, it is usually convenient to consider the membrane solution of the shell as being a particular solution of the more complicated and higher-order differential equations governing the bending theory.

The membrane solution of a spherical shell under uniform loading, q, gives the direct stress resultants N_x and N_y as:

$$N_x = N_y = \tfrac{1}{2} q a \tag{1.30}$$

The modified stress function $\bar{\bar{\Phi}}$ must therefore be given by a function of the form:

$$\bar{\bar{\Phi}} = \tfrac{1}{4} q \alpha^2 (x^2 + y^2) \tag{1.31}$$

in order to satisfy 1.30. Substitution of $\bar{\bar{\Phi}}$ from 1.31. into the governing partial differential equations 1.1 shows that this function does indeed represent a particular solution; and the vector V_0 is therefore given by the following relation:

$$\Big\{V_0(z)\Big\} = q \begin{Bmatrix} \tfrac{1}{4}\alpha^2(x^2+y^2) \\ 0 \end{Bmatrix}. \tag{1.32}$$

The kernel of the integral equation 1.29 is given by:

$$K(t_0,t) = A_{0,0}\, K_0(t_0,t) + A_{2,0} \frac{\partial^2}{\partial \xi^2} K_0(t_0,t) + A_{1,1} \frac{\partial^2}{\partial \xi \partial \eta} K_0(t_0,t) + $$
$$+ A_{0,2} \frac{\partial^2}{\partial \eta^2} K_0(t_0,t) \tag{1.33}$$

The term $A(t_0)$ of the equation 1.29 is obtained from the general representation 1.22 and may be reduced to a constant matrix given by:

$$A(t_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pi/a & 0 & \pi/a & 0 \\ 0 & \pi/a & 0 & \pi/a \end{bmatrix} \tag{1.34}$$

The (4 x 1) vector on the R.H.S. of 1.29 obtained from the general formula 1.22 is:

$$\left\{ f(t_0) \right\} = q \begin{Bmatrix} -\frac{\alpha^2}{4}(\xi^2+\eta^2) \\ 0 \\ -\alpha^2/2 \\ 0 \end{Bmatrix}. \tag{1.35}$$

The integrals appearing in the kernel function $K(t_0, t)$ are evaluated by a multi-step Simpson's Rule procedure. The convergence of this integral is very good and no difficulties are encountered in its evaluation, since the integrands are all regular.

The solution of the singular integral equation 1.29 is obtained by a numerical evaluation of the integral in terms of the unknown vector functions μ_i at points t_i around the contour. This leads to a system of (4 x n) linear algebraic equations for μ_i; where n is the number of interval points used in the integration. Numerical difficulties arising at the singularity in the kernel $K(t_0, t)$ may be overcome by a special technique of expansion of the unknown function μ in a Taylor Series about the singular point [3]. The results discussed in the following paragraph refer to a 4 and 6 step Simpson's Rule integration around the contour with a Taylor Series truncated after the second term. The integrals in the kernel were evaluated by a 40 step Simpson's rule. The deflection and stress function at any point in the shell are then determined by substituting the now known vector μ into the general solution 1.16.

A comparison with an exact solution of the governing differential equations for a circular plan form shallow spherical shell shows very good agreement, considering the small number or steps used in the numerical integration of equation 1.29. The calculated normal deflections of the shell for 4 and 6 step contour integration are shown for comparison with the exact values in the following table:

TABLE 1

Radial Coord.	$\overline{w}$ 4 step Int.	$\overline{w}$ 6 step Int.	$\overline{w}$ Exact.
0	0.433	0.543	0.541
0.4	0.447	0.540	0.519
0.8	0.461	0.557	0.537
1.2	0.480	0.582	0.561
1.6	0.487	0.591	0.572
2.0	0.434	0.528	0.512
2.4	0.258	0.316	0.300
2.6	0.116	0.126	0.123

The maximum discrepancy between the exact deflections and the deflections obtained from a 6 step contour integration is 5%. The agreement is considerably improved in regions immediately adjacent to the shell boundary, which are most important for bending stress calculations.

The stress function $\overline{\Phi}$ calculated from the 6 step integration is also within 5% of the exact values. Application of Richardson's Extrapolation to the stress function values from 4 and 6 step contour integration will improve the agreement to less than 1%, as shown in table 2.

Further accuracy may be obtained, if desired, by the use of a more accurate integration formula in the contour integration and retaining more terms in the Taylor Series.

TABLE 2.

Radial Coord.	$\bar{\Phi}$ Exact.	Φ Calculated.
0.4	-13.821	-13.849
0.8	-13.819	-13.842
1.2	-13.799	-13.799
1.6	-13.741	-13.692
2.0	-13.647	-13.525
2.4	-13.592	-13.422

PART 2

1. *Boundary-Value Problems for Shallow Circular Cylindrical Shells*

The shallow circular cylindrical shell equations obtained from [1] are of the form:

$$\left.\begin{aligned} &\Delta\,\Delta(\Phi) - \frac{Eh}{r}\,\frac{\partial^2 w}{\partial x^2} = 0, \\ &\frac{Eh^3}{12(1-\nu^2)}\,\Delta\,\Delta\,(w) + \frac{1}{r}\,\frac{\partial^2\Phi}{\partial x^2} = q. \end{aligned}\right\} \tag{2.1}$$

As in the previous case of the spherical shell, it is convenient to define new unknowns corresponding to the relations 1.2. Equations 2.1 may then be written in matrix form as:

$$\Delta\,\Delta\begin{Bmatrix}\bar{\Phi}\\ \bar{w}\end{Bmatrix} - \begin{bmatrix}0 & \alpha^2\\ -\alpha^2 & 0\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{Bmatrix}\bar{\Phi}\\ \bar{w}\end{Bmatrix} = \begin{Bmatrix}0\\ \alpha^4 q\end{Bmatrix}. \tag{2.2}$$

The L.H.S. of the above matrix partial differential equation 2.2 may be factorised, and the equation written in the form:

$$\left(\Delta\begin{bmatrix}+1 & 0\\ 0 & +1\end{bmatrix} + 2\,[K]\frac{\partial}{\partial x}\right)\left(\Delta\begin{bmatrix}+1 & 0\\ 0 & +1\end{bmatrix} - 2\,[K]\frac{\partial}{\partial x}\right)\begin{Bmatrix}\bar{\Phi}\\ \bar{w}\end{Bmatrix} = \begin{Bmatrix}0\\ \alpha^4 q\end{Bmatrix}, \tag{2.3}$$

Where K is a (2 x 2) matrix satisfying the relation:

$$[K]^2 = \frac{1}{4}\begin{bmatrix}0 & \alpha^2\\ -\alpha^2 & 0\end{bmatrix}. \tag{2.4}$$

It follows immediately that the general solution of the matrix equation 2.3 may be represented in the form:

$$\begin{Bmatrix}\bar{\Phi}\\ \bar{w}\end{Bmatrix} = \{V_0\} + \{V_1\} + \{V_2\}, \tag{2.5}$$

Where V_0 is a (2 x 1) particular solution vector; V_1 and V_2 are (2 x 1) function vectors satisfying the following homogeneous matrix partial differential equations:

$$\Delta\left\{V_1\right\} + 2\left[K\right]\frac{\partial}{\partial x}\left\{V_1\right\} = 0,$$

$$\Delta\left\{V_2\right\} - 2\left[K\right]\frac{\partial}{\partial x}\left\{V_2\right\} = 0, \tag{2.6}$$

Consider now the 2nd order equation 2.6.1; substituting for the vector V_1 a function defined by:

$$\left\{V_1\right\} = e^{-[k]x}\left\{\psi_1\right\} \tag{2.7}$$

equation 2.6.1 becomes:

$$\Delta\left\{\psi_1\right\} - \left[K\right]^2\left\{\psi_1\right\} = 0 \tag{2.8}$$

Similarly, substitution of a function defined by:

$$\left\{V_2\right\} = e^{[k]x}\left\{\psi_2\right\} \tag{2.9}$$

into equation 2.6.2 yields an equation for ψ_2 identical in form to equation 2.8.

The equation 2.8 is exactly analogous to equation 1.8 for V_2 considered in part 1 of this paper. Following the procedure already outlined there, the general solution of 2.8 in an arbitrary simply connected domain may be represented in complex notation as:

$$\left\{\psi_{1,2}(z)\right\} = \mathrm{Re}\left(\left[G(z,o,z,\bar{z})\right]\left\{\phi_{1,2}(z)\right\} - \int_0^z \frac{\partial}{\partial\sigma}\left[G(\sigma,o,z,\bar{z})\right]\left\{\phi_{1,2}(\sigma)\right\}d\sigma\right) \tag{2.10}$$

in which $\phi_{1,2}$ are arbitrary (2 x 1) analytic vectors, and $G(z,\zeta,t,\tau)$ is the Riemann Function associated with equation 2.8 and defined by:

$$\left[G(z,\zeta,t,\tau)\right] = J_0\left(\sqrt{\frac{1}{4}\begin{bmatrix} 0 & \alpha^2 \\ -\alpha^2 & 0\end{bmatrix}(z-t)(\zeta-\tau)}\right) \tag{2.11}$$

The general solution of the shallow circular cylindrical shell equations may therefore be written from 2.5, 2.7 and 2.9 as:

$$\begin{Bmatrix}\bar{\Phi}(z) \\ \bar{w}(z)\end{Bmatrix} = \left\{V_0(z)\right\} + e^{-[k]x}\left\{\psi_1(z)\right\} + e^{[k]x}\left\{\psi_2(z)\right\} \tag{2.12}$$

Substituting for ψ_1 and ψ_2 from 2.10, and combining the (2 x 1) vectors ϕ_1 and ϕ_2 to form a (4 x 1) vector ϕ, the solution 2.12 may be rewritten as:

$$\begin{Bmatrix}\bar{\Phi}(z) \\ \bar{w}(z)\end{Bmatrix} = \left\{V_0(z)\right\} + \left[e^{[k]x} e^{-[k]x}\right]\mathrm{Re}\left(\begin{bmatrix}G(z,o,z,\bar{z}) & o \\ o & G(z,o,z,\bar{z})\end{bmatrix}\left\{\phi(z)\right\}\right.$$

$$\left. - \int_0^z \frac{\partial}{\partial\sigma}\begin{bmatrix}G(\sigma,o,z,\bar{z}) & o \\ o & G(\sigma,o,z,\bar{z})\end{bmatrix}\left\{\phi(\sigma)\right\}d\sigma\right) \tag{2.13}$$

The generalised procedure for the determination of the arbitrary complex vector ϕ in the solution 2.13 from the general boundary conditions

1.17 follows exactly the procedure outlined for spherical shells. A typical application is given in the following example.

2. *Example Calculation of a Shell, Square in Plan with Rounded Corners.*

The shell is assumed to be characterised by the following parameters: $h = 0.036$ ins., $r = 12$ ins., $\nu = 0.3$

The boundary of the shell may be represented by the following parametric equation (6):

$$\left.\begin{aligned} x &= \frac{25a}{48}\left(\cos\theta - \frac{\cos 5\theta}{25}\right), \\ y &= \frac{25a}{48}\left(\sin\theta - \frac{\sin 5\theta}{25}\right). \end{aligned}\right\} \tag{2.14}$$

When a is the projected length of side of the square plan, and is taken as 10 ins.

The boundary conditions are assumed to be:

$$\bar{\Phi} = \bar{w} = \frac{\partial^2\bar{\Phi}}{\partial n^2} = \frac{\partial^2\bar{w}}{\partial n^2} = 0, \tag{2.15}$$

Where n is the normal to the shell boundary contour L. Using the parametric equation 2.14, the boundary conditions 2.15 may be rewritten in matrix form using the cartesian coordinates (x, y) as:

$$\begin{bmatrix} +1 & 0 \\ 0 & +1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\Phi} \\ \bar{w} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \cos^2\gamma & 0 \\ 0 & \cos^2\gamma \end{bmatrix} \frac{\partial^2}{\partial x^2}\begin{Bmatrix} \bar{\Phi} \\ \bar{w} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sin 2\gamma & \\ 0 & \sin 2\gamma \end{bmatrix} \frac{\partial^2}{\partial x \partial y}\begin{Bmatrix} \bar{\Phi} \\ \bar{w} \end{Bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sin^2\gamma & 0 \\ 0 & \sin^2\gamma \end{bmatrix} \frac{\partial^2}{\partial y^2}\begin{Bmatrix} \bar{\Phi} \\ \bar{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2.16}$$

Where γ is the angle given by:

$$\left.\begin{aligned} \gamma &= \tan^{-1}\left(\frac{dy}{dx}\right) + \pi/2 \\ \frac{dy}{dx} &= \frac{4}{625}\left(\frac{\sin 4\theta}{\sin^2\theta - \frac{6\sin\theta\sin 5\theta}{25} + \frac{\sin^2 5\theta}{125}} - \frac{x}{y}\right) \end{aligned}\right\} \tag{2.17}$$

Following a similar procedure to that described in part 1, contour integral representations of the arbitrary analytic vector ϕ consistent with the form of the boundary conditions 2.16 are introduced as follows:

$$\left\{\phi(z)\right\} = \int_L \left(1 - \frac{z}{t}\right) \lg\left(1 - \frac{z}{t}\right) \left\{\mu(t)\right\} ds + \int_L \left\{\mu(t)\right\} ds \tag{2.18}$$

The contour L in this example being the square with rounded corners defined by the parametric equation 2.14.

Substituting for ϕ from 2.18 into the relation 2.13, the following final form of the general solution is obtained:

$$\begin{Bmatrix} \bar{\Phi}(z) \\ \bar{w}(z) \end{Bmatrix} = \left\{ V_0(z) \right\} + \left[e^{[k]x} e^{-[k]x} \right] \int_L \left[K_0(z,t) \right] \left\{ \mu(t) \right\} ds,$$

Where $K_0(z,t)$ is a (4 x 4) matrix defined by:

$$K_0(z,t) = \begin{bmatrix} G(o,o,z,\bar{z}) & o \\ o & G(o,o,z,\bar{z}) \end{bmatrix} + \mathrm{Re}\left(\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \left(1-\frac{z}{t}\right) \lg \left(1-\frac{z}{t}\right) \right.$$
$$\left. - \int_0^z \frac{\partial}{\partial\sigma} \begin{bmatrix} G(\sigma,o,z,\bar{z}) & o \\ o & G(\sigma,o,z,\bar{z}) \end{bmatrix} \left(1-\frac{\sigma}{t}\right) \lg \left(1-\frac{\sigma}{t}\right) d\sigma \right). \qquad (2.20)$$

Substituting of the general solution 2.19 into the boundary conditions 2.16 yields a real singular integral equation for the unknown (4 x 1) real vector function μ. This procedure is exactly analogous to that already described in Part 1 of this paper. The solution of this integral equation yields the values of μ which may then be substituted back into the general solution 2.19. The stresses and deflections at any point in the shell then be computed as required.

This example was solved for the case of a unit normal load q. The solution procedure was similar to that described in the previous example. Graphs of deflection and the boundary component of the stress function are given as figs. 1 and 2.

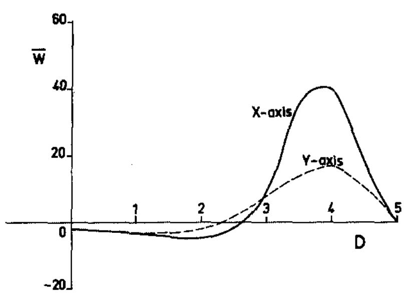

Fig. 1 Normal Deflection, $\bar{W}$./Distance from Centre Along Axes, D (ins)

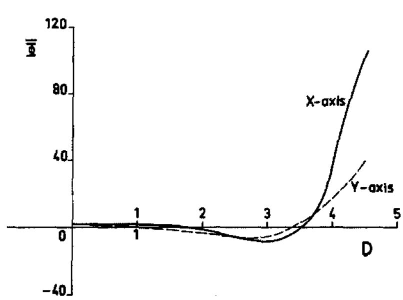

Fig. 2 Boundary Component of Modified Stress Function. $\bar{\Phi}$/Distance from Centre Along Axes, D (ins)

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